Uitwerking tentamen
Kwantumfysica 1
19 APRIL 2005

Problem 1

a) $\lambda_{Broglie} = \frac{h}{P_x}$, $P_x = \sqrt{2 m_e E_{kin}}$

$\langle E_{kin} \rangle = 200\ eV \Rightarrow$

$\langle \lambda_{Broglie} \rangle \approx \frac{h}{\sqrt{2 m_e \langle E_{kin} \rangle}}$

$= \frac{6.626 \cdot 10^{-34}}{\sqrt{2 \cdot 9.1 \cdot 10^{-31} \cdot 200 \cdot 1.602\ J/eV}} = 8.7 \cdot 10^{-11}\ m$

b) $\Delta E_{kin} \cdot \Delta t \approx \frac{\hbar}{2}$ for the generated pulses $\Rightarrow$

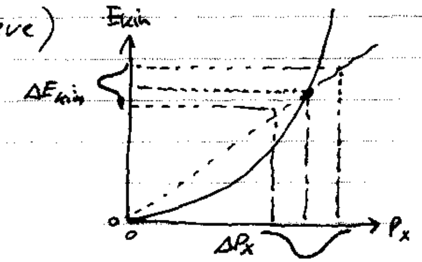
$\Delta E_{kin} \approx \frac{\hbar}{2 \Delta t}$ $\Delta t = 100\ ns$ $\Rightarrow$

$\Delta E_{kin} \approx \frac{1.05 \cdot 10^{-34}}{2 \cdot 100 \cdot 10^{-9}} \cdot \left(\frac{1}{1.602 \cdot 10^{-19}} \frac{eV}{J} \right) = 3.3 \cdot 10^{-9}\ eV$

$\Delta E_{kin} \ll \langle E_{kin} \rangle \Rightarrow$ linearizing $E_{kin} - P_x$ relation is Ok

Method 1 (good enough here)

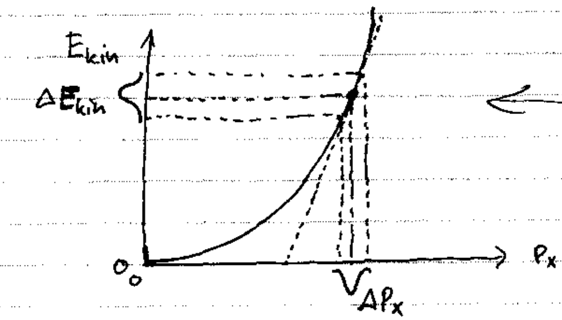
$\frac{\Delta E_{kin}}{\langle E_{kin} \rangle} \approx \frac{\Delta P_x}{\langle P_x \rangle}$



$\langle P_x \rangle = \sqrt{2 m_e \langle E_{kin} \rangle} = 7.6 \cdot 10^{-24}\ kg\ m/s$

$\Delta P_x \approx \langle P_x \rangle \cdot \frac{\Delta E_{kin}}{E_{kin}} = 7.6 \cdot 10^{-24} \cdot \frac{3.3 \cdot 10^{-9}}{200} = 1.2 \cdot 10^{-34}\ kg\ m/s$

Method 2 (a bit better)



$P_x = \sqrt{2 m_e E_{kin}} \Rightarrow \Delta P_x \approx \left. \frac{d\left(\sqrt{2 m_e E_{kin}}\right)}{d E_{kin}} \right|_{at \langle E_{kin} \rangle} \cdot \Delta E_{kin}$

$\Rightarrow \Delta P_x \approx \frac{1}{2} \sqrt{\frac{2 m_e}{\langle E_{kin} \rangle}} \cdot \Delta E_{kin}$

$= \frac{1}{2} \sqrt{\frac{2 \cdot 9.1 \cdot 10^{-31}}{200 \cdot 1.602 \cdot 10^{-19}\ J/eV}} \cdot 3.3 \cdot 10^{-9} \cdot 1.602 \cdot 10^{-19}\ J/eV$

$= 0.6 \cdot 10^{-34}\ kg\ m/s$

c) Electron pulses are emitted with a Gaussian time-profile, so they also have a Gaussian profile in space, because the uncertainty $\Delta P_x \ll \langle P_x \rangle$ (During the pulse, electrons are emitted with a constant velocity v_x)

This gives a FWHM pulse width **Δx** at the screen of

$$\Delta x \approx v_x \cdot \Delta t = \frac{\langle p_x \rangle}{m_e} \cdot \Delta t = \frac{7.6 \cdot 10^{-24}}{9.1 \cdot 10^{-31}} \cdot 100 \cdot 10^{-9} = 0.83\ m$$

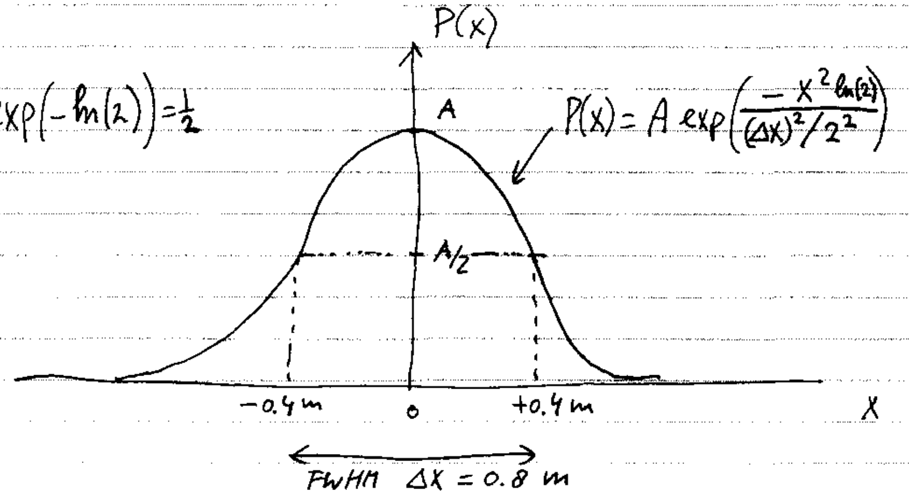
Not needed in your answer

Note 1) This value Δx concerns the probability density $P(x) = |\Psi(x)|^2$

2) During the time of flight from source to screen, another contribution to Δx increased in time. This contribution results from the uncertainty Δp_x, and adds an amount of about $T_{fly} \cdot \Delta p_x$. With $T_{fly} \approx 0.36\ \mu s$, this gives about $10^{-40}\ m \Rightarrow$ It can be neglected.

3) The product $\Delta x \cdot \Delta p_x$ for this state is $0.83\ m \cdot 0.63 \cdot 10^{-34} \approx \frac{\hbar}{2}$.

$\exp(-\ln(2)) = \frac{1}{2}$

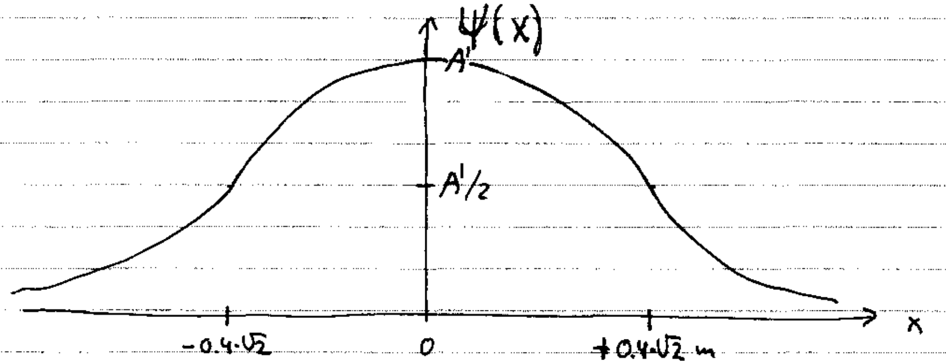


Assume $\Psi(x)$ a real and positive Gaussian

$$\Psi(x) = \sqrt{P(x)}\ , \quad A' = \sqrt{A}$$

$$\Psi(x) = A' \exp\left(\frac{-x^2 \ln(2)}{(\Delta x)^2/2^2}\right)^{1/2}$$

$$= A' \exp\left(\frac{-x^2 \ln(2)}{(\Delta x \cdot \sqrt{2})^2/2^2}\right)$$



This state is normalized for

$$A' = \frac{1}{a^{1/2}\, \pi^{1/4}} \quad \text{(see book p. 56)}$$

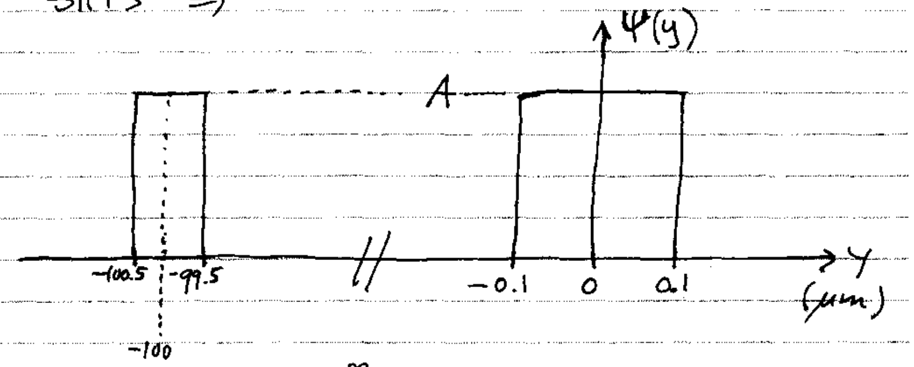
with $2a^2 = \frac{(\Delta x \sqrt{2})^2}{2^2 \ln(2)} \Rightarrow a = \frac{\Delta x}{2\sqrt{\ln(2)}} \Rightarrow$

$$A' = 1.05\ m^{-\frac{1}{2}}$$

Note: also correct is a rough guess that $A' \simeq 1$, and using Δx as standard deviation instead as FWHM value.

5

d) Thin screen with sharp edges of the slits ⇒



Normalized if $\int_{-\infty}^{\infty} |\psi(y)|^2 dy = 1$ ⇒

$A^2 \cdot 0.3\,\mu m = 1 \Rightarrow A = \sqrt{\frac{1}{0.3\,\mu m}}$

e) k_y-representation of state of right slit in d)

⇒ Use Fourier transform relation on normalized state for electrons coming out of the right slit ⇒

$$\psi_R(y) = \begin{cases} 0, & |y| > a_R/2 \\ A_R, & |y| \le a_R/2 \end{cases}$$

with $a_R = 0.2\,\mu m$ and

$A_R = \sqrt{\frac{1}{0.2\,\mu m}}$ (normalized as in d) for right slit)

$$\bar{\Psi}_R(k_y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_R(y)\, e^{-ik_y y}\, dy$$

also book p. 122

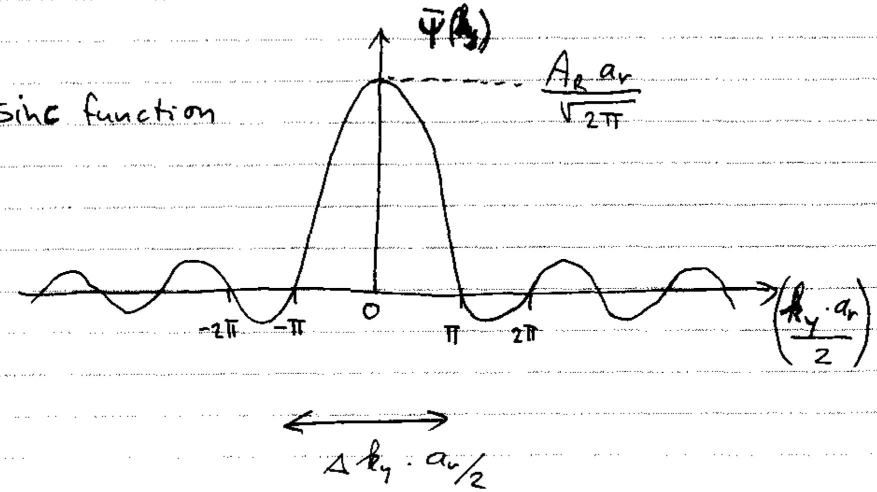
$$= \frac{1}{\sqrt{2\pi}} \int_{-a_R/2}^{a_R/2} A_R\, e^{-ik_y y}\, dy$$

$$= \frac{-A_R}{\sqrt{2\pi}\, ik_y} \left[e^{-ik_y y} \right]_{-a_R/2}^{a_R/2}$$

$$= \frac{-A_R}{\sqrt{2\pi}\, ik_y} \left(e^{-\frac{ik_y a_R}{2}} - e^{+\frac{ik_y a_R}{2}} \right)$$

$$= \frac{A_R\, a_R}{\sqrt{2\pi}} \left(\frac{\sin(k_y a_R/2)}{(k_y a_R/2)} \right)$$

sinc function



Uncertainty Δy from plot of d)
(right slit) $\Rightarrow$ $\Delta y \cong \frac{a_R}{\sqrt{12}} \approx 58$ nm
(any estimate close to 100 nm is good, but see hoor college of week 2, the 2nd)

Uncertainty Δp_y : from plot of $\overline{\Psi}(k_y)$

estimate $\Delta k_y \cdot a_r/2 \approx 2\pi$
$\Delta p_y = \hbar \cdot \Delta k_y$ $\Bigg\} \Rightarrow$

$\Delta p_y \approx \hbar \cdot \frac{4\pi}{a_r} \Rightarrow$

$\Delta p_y \cdot \Delta y \approx \hbar \cdot \frac{4\pi}{a_R} \cdot \frac{a_R}{\sqrt{12}} = \hbar \frac{4\pi}{\sqrt{12}} \approx 3.6\, \hbar$

$\Rightarrow$ Not so much in excess of $\hbar/2$, which is minimum according to Heisenberg

f) For the right beam after the screen,
$\Delta p_{y_R} \approx \hbar \cdot \frac{4\pi}{a_R} \approx 0.66 \cdot 10^{-26}$ kg m/s

For the left beam after the screen,
$\Delta p_{y_L} \approx \hbar \cdot \frac{4\pi}{a_L} \approx 1.32 \cdot 10^{-26}$ kg m/s

The time of flight to the detector is distance/speed $= \frac{2 \text{ meter}}{\langle p_x \rangle / m_e}$

$T_{fly2} = 0.23\ \mu s$

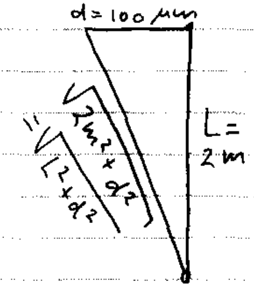
During this time, the uncertainties Δy_R and Δy_L increased to about

$\Delta y_R = \frac{\Delta p_{yR}}{m_e} \cdot T_{fly2} \approx 1.7$ mm

$\Delta y_L = \frac{\Delta p_{yL}}{m_e} \cdot T_{fly2} \approx 3.4$ mm

This is larger than the distance d between the slits, so the beams overlap.

g)



For φ controlled at φ=0, electrons from the left beam have a longer path length to the detector.

This path length difference
$\Delta L = \sqrt{L^2+d^2} - L$.

This gives an offset to the interference pattern of $\varphi_{off} = 2\pi \cdot \frac{\Delta L}{\lambda_{Broglie}} \approx 46.1\,\pi$ is equivalent to $0.1\,\pi$

~~The~~ The interference results from adding the amplitude of the left and right partial wave functions at the detector. Since the left slit is twice as narrow as the right slit, it spreads out in y direction twice as fast (see f). So, at the detector, the right partial wave has an amplitude that is two times that of the left wave.

⇒ Amplitudes at the detector:

Left partial wave $\psi_{left} \propto A e^{i(\varphi_{off}+\varphi)}$

Right partial wave $\psi_{right} \propto 2A$

Say $\varphi_t = \varphi_{off} + \varphi$, A some real constant

Detected electron intensity $P(\varphi_t)$ is

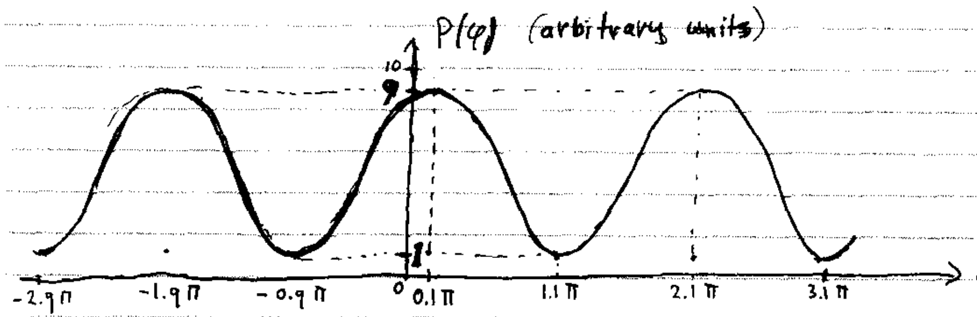
$$P(\varphi_t) = |\psi_{left} + \psi_{right}|^2$$

$$\propto |Ae^{i\varphi_t} + 2A|^2 = (Ae^{i\varphi_t}+2A)^*(Ae^{i\varphi_t}+2A)$$

$$(Ae^{-i\varphi_t} + 2A)(Ae^{i\varphi_t} + 2A) =$$

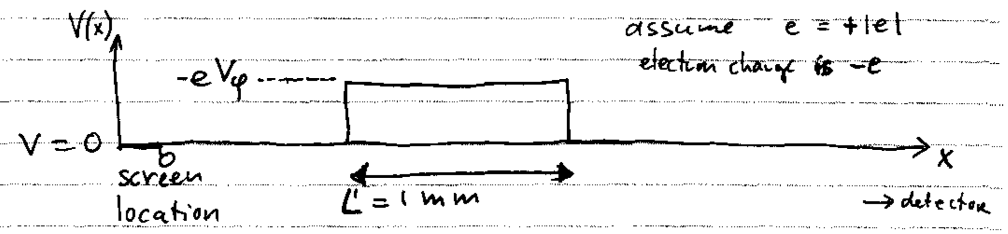
$$A^2(1 + 4 + 4\cos\varphi_t) \propto 5 + 4\cos(\varphi_t)$$

↳ $\varphi_{off} + \varphi$



h) Yes, see hoor college week 1, the 2nd lecture

i) Potential landscape seen by electrons in the left beam:



When one controls V_φ (here shown for $V_\varphi < 0$), the kinetic energy of the electron changes when it travels between the plates. The phase accumulated while crossing the plate region is

$$\varphi_{total} = 2\pi \frac{L'}{\lambda_{Broglie}} = L' \cdot k_x$$

To control a difference of π, the value of k_{x0} should be changed to k_{xV} such that

$$L'(k_{x0} - k_{xV}) = \pi$$

$$k_{x0} = \frac{1}{\hbar}\sqrt{2 m_e E_{kin_{200eV}}}$$

$$k_{xV} = \frac{1}{\hbar}\sqrt{2 m_e (E_{kin_{200eV}} - eV_\varphi)}$$

$$\Rightarrow k_{xV} = \frac{L k_{x0} - \pi}{L}$$

$$\Rightarrow eV_{\varphi=\pi} = -\left(\frac{\hbar^2 k_{xV}^2}{2m_e} - E_{kin_{200eV}}\right)$$

$$V_{\varphi=\pi} = \frac{1}{e}\Big(\underset{20eV}{E_{kin}} - \frac{\hbar^2}{2m_e}\left(\frac{L k_{x0} - \pi}{L}\right)^2\Big)$$

Filling in gives $V_{\varphi=\pi} = 17\ \mu V$

The experiment will work, but it will be very sensitive to noise and stray fields, because 17 μV is in practice a low voltage.

Problem 2

See book p. 80, 81 eqs. (3.45) – (3.49)

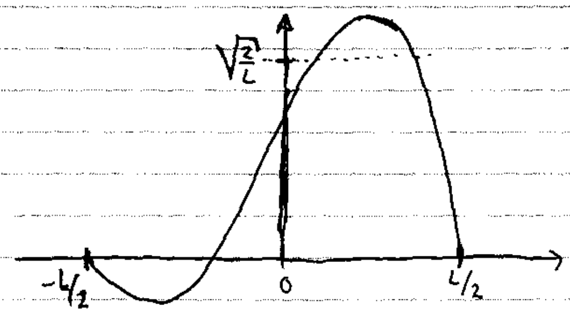
Problem 3

a) See book p. 93 and p. 179 eq. (6.100)

$\varphi_1(x) = \sqrt{\frac{2}{L}} \cos\left(\frac{\pi x}{L}\right)$, $|x| \le \frac{L}{2}$, $\varphi_1(x) = 0$ elsewhere

$\varphi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$, $|x| \le \frac{L}{2}$, $\varphi_2(x) = 0$ elsewhere

b)



c) $\psi(x,0) = \varphi_1 \rightarrow |\varphi_1\rangle$

i) $P(a_1) = |\langle \varphi_1 | \varphi_1 \rangle|^2 = 1$

ii) $P(a_2) = |\langle \varphi_2 | \varphi_1 \rangle|^2 = 0$

iii) $P(E_1) = |\langle \varphi_1 | \varphi_1 \rangle|^2 = \left|\langle \varphi_1 | \left(\frac{|\varphi_1\rangle + |\varphi_2\rangle}{\sqrt{2}}\right)\right|^2 = \frac{1}{2}$

iv) $P(E_2) = |\langle \varphi_2 | \varphi_1 \rangle|^2 = \left|\langle \varphi_2 | \left(\frac{|\varphi_1\rangle + |\varphi_2\rangle}{\sqrt{2}}\right)\right|^2 = \frac{1}{2}$

d) $\hat{U} = e^{-i\hat{H}t/\hbar}$

$|\psi(t)\rangle = \hat{U}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_1 t/\hbar}|\varphi_1\rangle + e^{-iE_2 t/\hbar}|\varphi_2\rangle\right) \Rightarrow$

$\psi(x,t) = \frac{1}{\sqrt{2}} e^{-iE_1 t/\hbar}\varphi_1(x) + \frac{1}{\sqrt{2}} e^{-iE_2 t/\hbar}\varphi_2(x)$

e)
$$\begin{cases} |\varphi_1\rangle = \frac{1}{\sqrt2}\left(|\psi_1\rangle + |\psi_2\rangle\right) \\ |\varphi_2\rangle = \frac{1}{\sqrt2}\left(|\psi_1\rangle - |\psi_2\rangle\right) \end{cases}$$

$$\begin{cases} \langle\varphi_1|\hat A|\varphi_1\rangle = \frac12\left(\langle\psi_1| + \langle\psi_2|\right)\hat A\left(|\psi_1\rangle + |\psi_2\rangle\right) = \frac{a_1+a_2}{2} \\ \langle\varphi_2|\hat A|\varphi_2\rangle = \frac12\left(\langle\psi_1| - \langle\psi_2|\right)\hat A\left(|\psi_1\rangle - |\psi_2\rangle\right) = \frac{a_1+a_2}{2} \\ \langle\varphi_1|\hat A|\varphi_2\rangle = \frac12\left(\langle\psi_1| + \langle\psi_2|\right)\hat A\left(|\psi_1\rangle - |\psi_2\rangle\right) = \frac{a_1-a_2}{2} \\ \langle\varphi_2|\hat A|\varphi_1\rangle = \frac12\left(\langle\psi_1| - \langle\psi_2|\right)\hat A\left(|\psi_1\rangle + |\psi_2\rangle\right) = \frac{a_1-a_2}{2} \end{cases}$$

$$\langle\psi(t)|\hat A|\psi(t)\rangle =$$

$$\frac12\left(e^{+iE_1t/\hbar}\langle\varphi_1| + e^{+iE_2t/\hbar}\langle\varphi_2|\right)\hat A\left(e^{-iE_1t/\hbar}|\varphi_1\rangle + e^{-iE_2t/\hbar}|\varphi_2\rangle\right) =$$

$$\frac12\langle\varphi_1|\hat A|\varphi_1\rangle + \frac12\langle\varphi_2|\hat A|\varphi_2\rangle + \frac12\langle\varphi_1|\hat A|\varphi_2\rangle e^{-i(E_2-E_1)t/\hbar} + \frac12\langle\varphi_2|\hat A|\varphi_1\rangle e^{+i(E_2-E_1)t/\hbar} =$$

$$\frac{a_1+a_2}{4} + \frac{a_1+a_2}{4} + \frac{a_1-a_2}{4}\left(2\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right) =$$

$$\left(\frac{a_1+a_2}{2}\right) + \left(\frac{a_1-a_2}{2}\right)\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)$$

f) No, because $\langle\hat A\rangle$ **depends** on time, and

$$0 \neq \frac{d\langle\hat A\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat H,\hat A]|\psi\rangle \Rightarrow [\hat H,\hat A] \neq 0$$

Can also be seen from the fact stated in the question itself that $\hat H$ and $\hat A$ do not have the same eigen functions.

g)
$$P\left(\substack{\text{ground}\\\text{state}}\right) = \left|\langle\varphi_{1\,New}|\psi(0)\rangle\right|^2$$

$$\langle\varphi_{1\,New}|\psi(0)\rangle = \int_{-\infty}^{\infty}\varphi^*_{1\,New}(x)\,\psi(x,0)\,dx$$

$$\varphi_{1,new}(x) = \begin{cases}\sqrt{\frac1L}\cos\left(\frac{\pi x}{2L}\right), & |x|\le L \\ 0, & |x| > L\end{cases}$$

$$\langle\varphi_{1,New}|\psi_0\rangle = \frac{1}{\sqrt2}\int_{-L/2}^{L/2}\sqrt{\frac1L}\cos\left(\frac{\pi x}{2L}\right)\left(\sqrt{\frac2L}\cos\left(\frac{2\pi x}{2L}\right) + \sqrt{\frac2L}\sin\left(\frac{2\pi x}{2L}\right)\right)dx$$

($\uparrow$ $\psi(x,0) = 0$ for $|x| > \frac L2$)

$$= \frac{1}{2L}\int_{-L/2}^{L/2}\cos\left(\frac{\pi x}{2L}\right) + \cos\left(\frac{3\pi x}{2L}\right) + \sin\left(\frac{3\pi x}{2L}\right) - \sin\left(\frac{3\pi x}{2L}\right)dx$$

$$= \frac{1}{2L}\left[-\frac{2L}{\pi}\sin\left(\frac{\pi x}{2L}\right) - \frac{2L}{3\pi}\sin\left(\frac{3\pi x}{2L}\right) + \frac{2L}{5\pi}\cos\left(\frac{5\pi x}{2L}\right) - \frac{2L}{3\pi}\cos\left(\frac{3\pi x}{2L}\right)\right]_{-L/2}^{L/2}$$

$$= \frac{-1}{\pi}\left(\sqrt2\right) - \frac{1}{3\pi}\left(\sqrt2\right) + \frac{1}{5\pi}(0) - \frac{1}{3\pi}(0)$$

$$= -\frac{4\sqrt2}{3\pi}$$

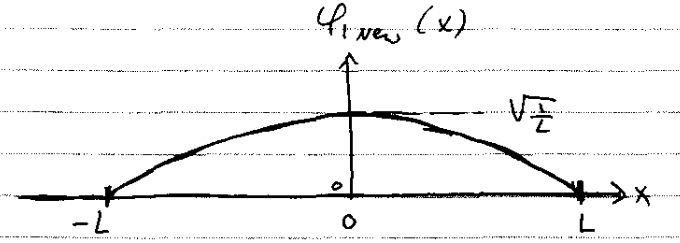
$$P\left(\substack{\text{New}\\\text{Ground}\\\text{State}}\right) = \left|\langle\varphi_{1\,New}|\psi(0)\rangle\right|^2 = \left(\frac{4\sqrt2}{3\pi}\right)^2 = \frac{32}{9\pi^2}$$

(h)

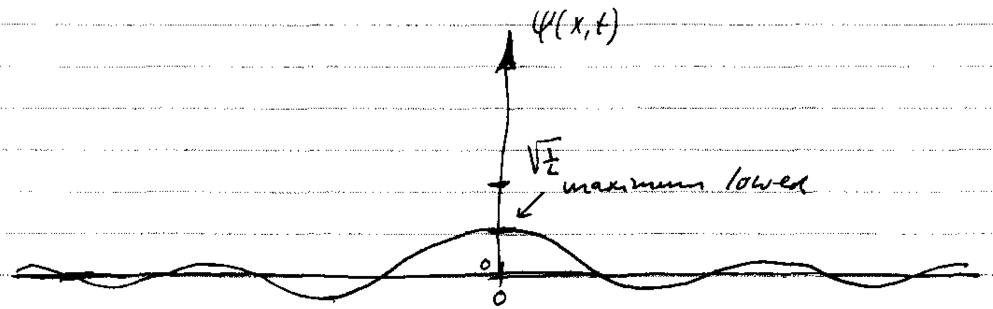
15

16

h) At $t = t_0$



At $t > t_0$ $\psi(x,t)$



1° $\langle x \rangle$ stays at $x = 0$, stays symmetric

2° Δx increases

3° Particle is free for $t > t_0$, so momentum distribution $(\bar{\Psi}(k_x, t))$ is fixed for $t > t_0 \Rightarrow$

$\bar{\Psi}(k_x, t)$ looks like a sinc function (see homework W5.1) $\Rightarrow$ causes structure in spreading x-wave packet.

END